1 Quantum Mechanics

EXERCISES

1.1 Show that a state ρ is pure iff $\mathrm{rank}(\rho) = 1$.

1.2 The set of all selfadjoint operators in $\mathbb{C}^2$ with unit trace is $\mathbb{R}^3$. The subset of the one-qubit states is the unit ball $B^3 := \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| \leq 1\}$. All points in the boundary $S^2 = \partial B^3$ represent pure states. Prove that this is no longer the case for quantum systems with a Hilbert space of dimension $N > 2$.

1.3 Let AB be a bipartite system. Many copies of the system in a state ρ are available. Alice carries measurements on A, and Bob on B. By exchanging information classically they can determine how their outcomes are correlated and thus they can measure the mean value of any observable of the form $X_A \otimes Y_B$. 1/ Prove that this is sufficient to know the expectation value of any observable of the compound system. 2/ Would this remain true if the Hilbert space was a real linear space instead of a complex vector space?

1.4 Prove that under arbitrary local unitary actions $U_A \otimes U_B$ the Schmidt number of $|\Psi\rangle_{AB}$ remains invariant. In other words, such local actions do not create entanglement.

1.5 A Schmidt decomposition for n-partite pure states would be of the form

$$|\Psi\rangle = \sum_j \sqrt{p_j} |j\rangle_1 \otimes |j\rangle_2 \otimes \ldots \otimes |j\rangle_n,$$

with $p_j > 0$, $\sum_j p_j = 1$, $\{|j\rangle_i\}$ an orthonormal basis in $\mathcal{H}_i$. Prove that such a decomposition does not exist in general for $n \geq 3$.

1.6 Let $|\Psi\rangle_{AB} := \sum_{i=1}^{d_A}\sum_{j=1}^{d_B}\gamma_{ij}|a_i\rangle|b_j\rangle \in \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, with $\{|a_i\rangle\}_1^{d_A}$, $\{|b_i\rangle\}_1^{d_B}$ orthonormal bases of $\mathcal{H}_A, \mathcal{H}_B$. Suppose that $d_A = d_B =: N$. Let $|\Psi\rangle_{AB} = \sum_{k=1}^{r}\sqrt{w_k}|u_k\rangle|v_k\rangle$, $w_k > 0$, $\sum_{k=1}^{r} w_k = 1$, be its Schmidt form. 1/ Prove that there exists a local unitary operator $U_A \otimes U_B$ which brings $|\Psi\rangle_{AB}$ into $\sum_{k=1}^{r}\sqrt{w_k}|a_k\rangle|b_k\rangle$. 2/ Compute the dimension of the set of orbits generated by generic vectors (i.e. with nonvanishing and simple weights) in Schmidt form under the action of the local unitary operators.

1.7 Let $\dim\mathcal{H}_A = \dim\mathcal{H}_B = \dim\mathcal{H}_C =: N$. By considering the dimensions of the orbits of generic Schmidt vectors in $\mathcal{H}_A\otimes\mathcal{H}_B\otimes\mathcal{H}_C$ under local unitaries $U_A\otimes U_B\otimes U_C$, prove that a generic pure vector of a tripartite system has not a Schmidt decomposition.

1.8 Consider the Werner state of a pair of qN-its

$$\rho_\Phi(\lambda) := \lambda|\Phi\rangle\langle\Phi| + (1-\lambda)\frac{1}{N^2}\mathbf{1},$$

where

$$|\Phi\rangle := \frac{1}{\sqrt{N}}\sum |j\rangle\otimes|j\rangle$$

is a maximally entangled state vector, $\{|j\rangle\}$ being an orthonormal basis in $\mathbb{C}^N$. Find for what values of λ is nonpositive the partial transpose of $\rho_\Phi(\lambda)$.

1.9 Consider the expansion channel $\mathcal{E} : \rho_A \mapsto \rho_A \otimes \rho_{\mathcal{B},0}$. Find a Kraus form for $\mathcal{E}$.

1.10 Prove that if $F = \{F_y : y \in Y\}$ is a POVM in a Hilbert space $\mathcal{H}$ of dimension $\dim\mathcal{H} =: N = |Y|$ such that all F_y are rank 1, then F is a von Neumann measurement.

1.11 Prove that a quantum operation $\mathcal{E}$ is invertible iff it is a unitary map.

1.12 1/ Prove that any quantum operation from a single qubit system to itself is of the form

$$\rho' := \mathcal{E}(\rho) = \sum_{\alpha\beta}\mathcal{E}_{\alpha\beta}\sigma_\alpha\rho\sigma_\beta,$$

where Greek indices run from 0 to 3, $\sigma_0 := \mathbf{1}_2$, and the coefficients $\mathcal{E}_{\alpha\beta}$ satisfy $\mathcal{E}_{\alpha\beta} = \mathcal{E}^*_{\beta\alpha}$. 2/ Which other conditions must $\mathcal{E}_{\alpha\beta}$ fulfill? 3/ Show that the polarizations $\mathbf{P}, \mathbf{P}'$ of ρ, ρ' are related by an affine transformation: $\mathbf{P}' = \mathbf{a} + M\mathbf{P}$, where M is a real matrix.

1.13 A qubit has a pure state ψ chosen randomly on the Bloch sphere. Estimate the average fidelity of a random guess ϕ.

1.14 As above, let qubit have a pure state $|\psi\rangle$ chosen randomly on the Bloch sphere. Suppose that we measure its spin along the z-axis. The resulting state is $\rho := p_+|+\rangle\langle+|+p_+|-\rangle\langle-|$, where $p_\pm := |\langle\pm|\psi\rangle|^2$. Calculate the average fidelity with which ρ represents P_ψ.

1.15 Solve the above two Exercises for a general qN-it.

1.16 Prove that the fidelity of two qubits states with polarization vectors $\mathbf{r}, \mathbf{s}$ is

$$F(\rho_{\mathbf{r}}, \rho_{\mathbf{s}}) = \frac{1}{2}\left(1 + \mathbf{r}\cdot\mathbf{s} + \sqrt{(1-r^2)(1-s^2)}\right).$$